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Approximation Of Cauchy Principal Value Integrals In Two Dimensions

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Abstract: Cubature rules have been formulated for the numerical approximation of real CPV integrals in two dimensions. The expressions for the error associated with the rules have been determined and numerical verification of the rules has been made using suitable examples.

Keywords: cpv integral, cubature rules, degree of precision, two dimensional.

1 INTRODUCTION

Cauchy Principal Value (CPV) integrals occur quite frequently in different branches of Applied Mathematics, Engineering and Physics. The quadrature rules derived for the numerical approximation of one dimensional CPV integrals given by

$$J(g) = P\int_{-\alpha}^{\alpha} \frac{g(x)}{x}\,dx \qquad (1)$$

can be found in the text due to Davis and Rabinowitz [1] and in the review article by Monegatto [2]. However, not sufficient research work has been done for the numerical approximation of two dimensional CPV integral which is given by

$$I(f) = P\iint_S \frac{f(x,y)}{xy}\,dxdy \qquad (2)$$

where S is a square with vertices at $(\pm\alpha,\pm\alpha)$ and f is continuous on S. The two dimensional CPV integral given by (2) can be defined by the following limit if it exists.

$$I(f) = \lim_{\delta\to 0}\iint_{S-s} \frac{f(x,y)}{xy}\,dxdy \qquad (3)$$

where $\delta > 0$ is an arbitrarily small positive number, s is a square with vertices at $(\pm\delta,\pm\delta)$.

It is noteworthy that Monegatto [3],Nayak, Acharya and Acharya [4] ,Theocaris et al [5], Theocaris and Kazantzakis [6] have considered the problem of numerical evaluation of two dimensional Cauchy Principal Value integral $I(f)$.

The objective in the present paper is to construct some cubature rules for the numerical evaluation of the two dimensional CPV integral $I(f)$ given by (2).

2 GENERATION OF AN EIGHT POINT RULE

We construct first an eight point rule for the numerical approximation of two dimensional CPV integral $I(f)$. Let the proposed 8-point rule be

$$\begin{aligned} R_8(f) = {} & C_1\left\{\sum\nolimits_1 f(\pm\alpha a,\pm\alpha a) - \sum\nolimits_2 f(\pm\alpha a,\pm\alpha a)\right\} \\ & + C_2\left\{\sum\nolimits_1 f(\pm\alpha b,\pm\alpha b) - \sum\nolimits_2 f(\pm\alpha b,\pm\alpha b)\right\} \end{aligned} \qquad (4)$$

where $a \neq b$, $\sum_1$ is the summation of function values for the arguments with same sign of parameter a and b and $\sum_2$ is the summation of function values for the arguments with opposite sign of parameter a and b .It is noteworthy that the rule is exact i.e. $I(f) = R_8(f)$ whenever $f(x,y)=x^n \times y^m$, $n+m$ is odd or both n, m are even. As the rule is also symmetric we make the rule exact for monomials $x^n \times y^m$ when (n,m)=(1,1),(3,1),(5,1) and get the following set of equations :

$$\left.\begin{aligned} C_1a^2 + C_2b^2 &= 1, \\ C_1a^4 + C_2b^4 &= 1/3, \\ C_1a^6 + C_2b^6 &= 1/5. \end{aligned}\right\} \qquad (5)$$

The solutions of the above system of equations treating a as a free parameter can be obtained in the following form:

$$\left.\begin{aligned} b &= \sqrt{\{(5a^2-3)/(15a^2-5)}, \\ C_1 &= (3b^2-1)/(3a^2b^2-3a^4), \\ C_2 &= (1-3a^2)/(3b^4-3a^2b^2). \end{aligned}\right\} \qquad (6)$$

where $a \neq b$ and $a \in (0,1/\sqrt{3}) \cup (\sqrt{3/5},1) = \Delta$ (say). The rule $R_8(f)$ is exact for all monomials $x^n \times y^m$ of degree ≤ 5 and also for the monomial x^5y.Therefore its degree of precision is five. Since the rule $R_8(f)$ involves only one parameter $a \in \Delta$, it is denoted as $R_8(f,a)$.

The truncation error associated with $R_8(f,a)$ is given by

$$E_8(f,a) = I(f) - R_8(f,a) \qquad (7)$$

The Taylors' series expansion of $f(x,y)$ about $(0,0)$ is given by

$$f(x,y) = \sum_{j=0}^{\infty} \frac{1}{j!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^j f(0,0) \qquad (8)$$

where all the partial derivatives are evaluated at (0,0). Using the Taylor series expansion in (7), the $E_8(f,a)$ is specified in the following form:

$$E_8(f,a) = \frac{-4\alpha^6 L}{405} + \frac{\alpha^8}{3375(3a^2-1)}\{(5a^4-1)N + \frac{(35a^4-30a^2+3)M}{49}\} + O(\alpha^{10}) \qquad (9)$$

where $L = f^{3,3}, M = f^{7,1} + f^{1,7}, N = f^{5,3} + f^{3,5}$, L is a partial derivative of order six and M and N are partial derivatives of order eight.

The rule $R_8(f,a)$ has degree of precision five. However, if the factor $C_R = -4\alpha^6 L/405$, the leading term in $E_8(f,a)$ is treated as a corrective factor and added to the approximation $R_8(f,a)$, then the resulting approximation will have degree of precision 7.

Since the corrective factor C_R is independent of a for the rule to be of simple form we take a =1 which implies b=1/√5.Using the corrective factor C_R the resulting approximation for the CPV integral $I(f)$ is given by

$$I(f) = R_8(f,1) + C_R = \overline{R}(f) \qquad (10)$$

and $\bar{E}(f)$ is the error associated with the rule $\bar{R}(f)$ is given by:

$$\overline{E}(f) = \frac{-4\alpha^8}{165375}\{f^{7,1} + f^{1,7}\} + \frac{-2\alpha^8}{3375}\{f^{5,3} + f^{3,5}\} + O(\alpha^{10}) \qquad (11)$$

Rule	$\lvert Error \rvert$
R_8	1.59×10^{-04}
$\bar{R}_8$	4.88×10^{-06}
R_{12}	7.23×10^{-10}

3 GENERATION OF TWELVE POINT RULE

We construct next a twelve point rule for the numerical evaluation of thse two dimensional CPV integral $I(f)$. Let the proposed 12-point rule be of the following form which includes partial derivatives f_x and f_y of the function $f(x,y)$ involved in the integrand of $I(f)$.

$$R_{12}(f) = A\{\sum\nolimits_1 f(\pm\alpha s, \pm\alpha s) - \sum\nolimits_2 f(\pm\alpha s, \pm\alpha s)\}$$
$$+ B\{\sum\nolimits_1 f(\pm\alpha t, \pm\alpha t) - \sum\nolimits_2 f(\pm\alpha t, \pm\alpha t)\}$$
$$+ \alpha C\{\sum\nolimits_1 f_x(0, \pm\alpha r) - \sum\nolimits_2 f_y(\pm\alpha r, 0)\} \qquad (12)$$

where s,t,r real numbers, A,B,C are coefficients and the notations $\sum_1$ and $\sum_2$ have the same meaning as in case of the eight point rule.

Following the same approach as in case of the eight point rule i.e. making $R_{12}(f)$ exact for all monomials in x and y of degree ≤ 8, we get

$$\left.\begin{array}{l} s = 0.805979782918599, t = 0.380554433208316, \\ r = \sqrt{6/7}, A = 0.365502660334805, \\ B = 3.594716818403184, C = 0.130681602560488. \end{array}\right\} \qquad (13)$$

Therefore the rule $R_{12}(f)$ has degree of precision nine if the (13) holds good.

Using the taylor series expansion for $f(x,y)$ in $I(f) - R_{12}(f)$, the truncation error $E_{12}(f)$ is given by:

$$E_{12}(f) = \frac{\alpha^{10}}{10!}\{K_1 U + K_2 V + K_3 W\} + O(\alpha^{12}) \qquad (14)$$

Where $U = f^{9,1} + f^{1,9}, V = f^{7,3} + f^{3,7}, W = f^{5,5}$ and $K_1 = 3.0697\times 10^{-1}, K_2 = 4.0882\times 10^{-2}$, $K_3 = -1.0034\times 10^{-2}$.

4 NUMERICAL VERIFICATION

For making the numerical verification the integral $J(\alpha) = \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{e^{x+y}}{xy}\,dxdy$ is considered where $\alpha = 0.5$.The exact value of the integral =1.028182817310825. The integral has been computed by $R_8(f)$ where a=1, the eight point rule $\bar{R}_8(f)$ with corrective factor and the twelve point rule $R_{12}(f)$. The computed values of magnitude of the error have been presented in the table.

TABLE

It is observed that the accuracy of the rules depends upon the factor.

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REFERENCES

[1] Davis, P.J. & Rabinowitz, P. *Methods of numerical integration* (2nd edn.), Academic Press, London, 1984.

[2] Monegato, G., *The numerical evaluation of one dimensional Cauchy principal value integrals,* computing, 29, 1982, 337-354.

[3] Monegato, G., *convergence of product formulas for the numerical evaluation of cetain two dimensional CPV integral,* Numer. Math., 43(2), 1984, 161-173.

[4] Nayak, M.M., Acharya, M. and Acharya, B.P., *Numerical evaluation of two dimensional Cauchy principal value integrals,* Appl. Math. and Comp., 225, 2013, 807-810.

[5] Theocaris,P.S. ,Kazantzakis, J.V. and N.I.Iokaimidis., *On the numerical evaluation of two dimensional principal value integrals*, Int. J. Numer. Meth. Engg., 15(4),1980,629-634.

[6] Theocaris, P.S. and Kazantzakis, J.V., *On the numerical evaluation of two and three dimensional Cauchy principal value integrals,* Acta Mech.,39,1981,105-115.

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